Convexity of 4-Maximal Neural Codes

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Introduction

Biologists have observed neurons in some animal brains called **place cells**, which act as position sensors. Place cells fire at high rates when the animal is inside corresponding regions of the environment called **place fields**. These fields have been observed to be approximately convex, and intersection patterns of place fields form neural codes. We investigate how the brain represents space by studying **convex neural codes** [2].

Neural Codes

A **neural code** or **code** on n neurons is a collection of firing patterns called **codewords**, $\mathcal{C} \subseteq 2^{[n]}$, where $\emptyset \in \mathcal{C}$. The **maximal codewords** of a code are its maximal elements with respect to set inclusion. We can **generate codes** using sets in Euclidean space.

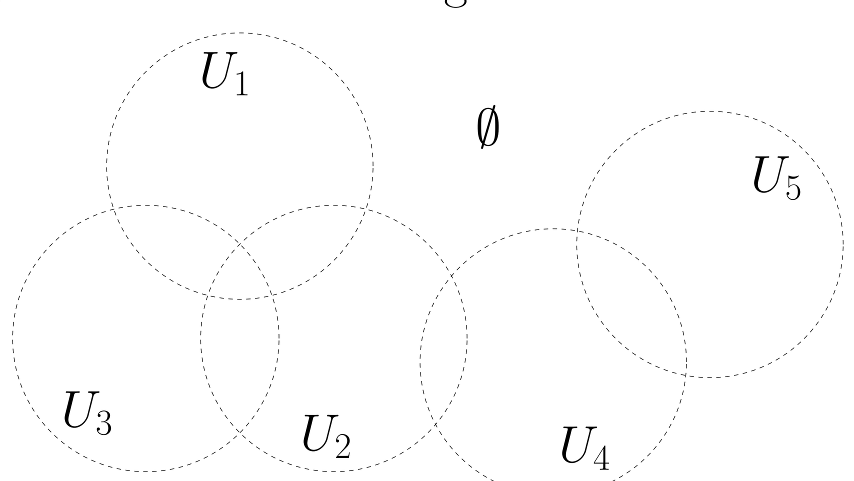


Figure 1: The code generated by $\{U_1, U_2, U_3, U_4, U_5\}$ is $\mathcal{C} = \{\mathbf{123}, \mathbf{24}, \mathbf{45}, 12, 13, 23, 1, 2, 3, 4, 5\}$

If $\mathcal{U}$ is composed by sets that are open and convex and $\mathcal{C}(\mathcal{U}) = \mathcal{C}$, then $\mathcal{C}$ is an **(open) convex code**.

Simplicial Complexes

A code $\mathcal{C}$ on n neurons has a **simplicial complex**

$$\Delta(\mathcal{C}) := \{\omega \subseteq [n] : \omega \subseteq \sigma \text{ for some } \sigma \in \mathcal{C}\}.$$

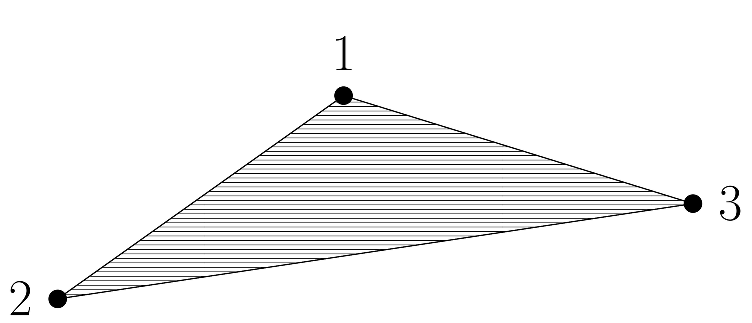


Figure 2: A geometric realization of a simplicial complex composed by one *2-simplex* (a filled-in triangle)

Elements of $\Delta(\mathcal{C})$ are called **faces** and maximal faces are called **facets**. Facets correspond exactly to the maximal codewords of $\mathcal{C}$.

Example: In Fig. 2, 12 is a face and **123** is a facet.

Characterizing Convexity

A code $\mathcal{C}$ is **max-∩-complete** if it contains every intersection of at least two facets of $\Delta(\mathcal{C})$. Max-∩-complete codes are convex, as shown in [1].

Consider a $\Delta(\mathcal{C})$ on n neurons and let $\sigma \subseteq [n]$. A **link of σ** with respect to $\Delta(\mathcal{C})$ is defined as

$$\mathrm{Lk}_{\Delta(\mathcal{C})}(\sigma) = \{\tau \subset [n] \setminus \sigma : \tau \cup \sigma \in \Delta(\mathcal{C})\}.$$

Example: For $\Delta(\mathcal{C}) = \{\mathbf{123}, \mathbf{124}, \mathbf{45}, 12, 13, 23, 14, 24, 1, 2, 3, 4, 5, \emptyset\}$, we look at the links $\mathrm{Lk}_{\Delta(\mathcal{C})}(1)$ and $\mathrm{Lk}_{\Delta(\mathcal{C})}(4)$:

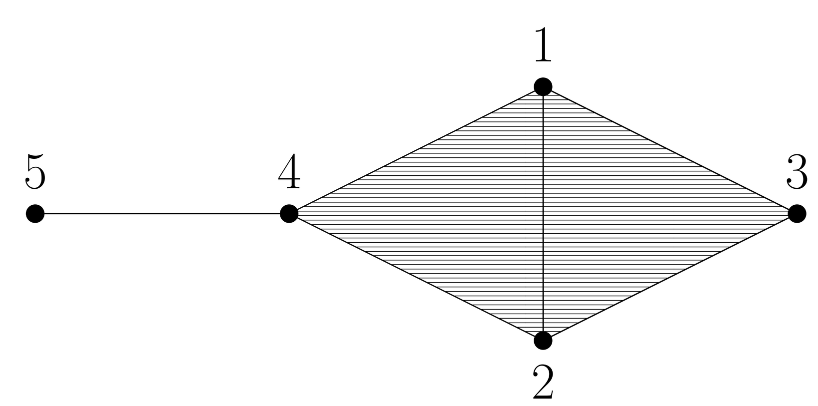


Figure 3: A geometric realization of $\Delta(\mathcal{C})$

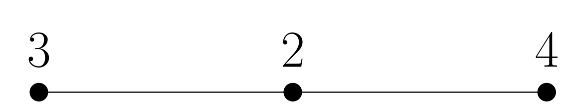


Figure 4: $\mathrm{Lk}_{\Delta(\mathcal{C})}(1) = \{23, 24, 2, 3, 4\}$

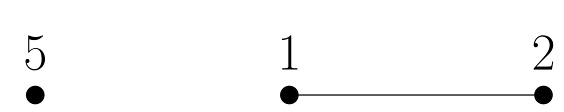


Figure 5: $\mathrm{Lk}_{\Delta(\mathcal{C})}(4) = \{12, 5, 1, 2\}$

Let $\mathcal{C}$ be a neural code with simplicial complex $\Delta(\mathcal{C})$. We say $\mathcal{C}$ has a **local obstruction at σ** if there exists some nonempty face $\sigma \in \Delta(\mathcal{C})$ such that:

- σ is the intersection of facets of $\Delta(\mathcal{C})$
- $\sigma \notin \mathcal{C}$
- $\mathrm{Lk}_{\Delta(\mathcal{C})}(\sigma)$ is not contractible.

It has been shown that codes with three maximal codewords are convex if and only if they have no local obstructions [4].

The Problem

Local obstructions do not appear in convex codes, as shown in [3]. Neither do *wheels*, a geometric and combinatorial object discussed in [6]. We investigate the converse for codes with four maximal codewords.

Conjecture

Let $\mathcal{C}$ be a neural code with 4 maximal codewords. If $\mathcal{C}$ has no local obstructions and no wheels, then $\mathcal{C}$ is convex.

Nerves

For a collection of subsets $\mathcal{W} = \{W_1, W_2, \cdots, W_n\}$ of a set X, the **nerve of** $\mathcal{W}$ is the simplicial complex

$$\mathcal{N}(\mathcal{W}) := \{I \subset [n] : \textstyle\bigcap_{i \in I} W_i \neq \emptyset\}.$$

A topological result called the *nerve lemma* [5] lets us investigate convexity by looking at the nerve of the set of facets $\mathcal{F}$ of a code $\mathcal{C}$, denoted by $\mathcal{N}(\mathcal{F})$.

Cases

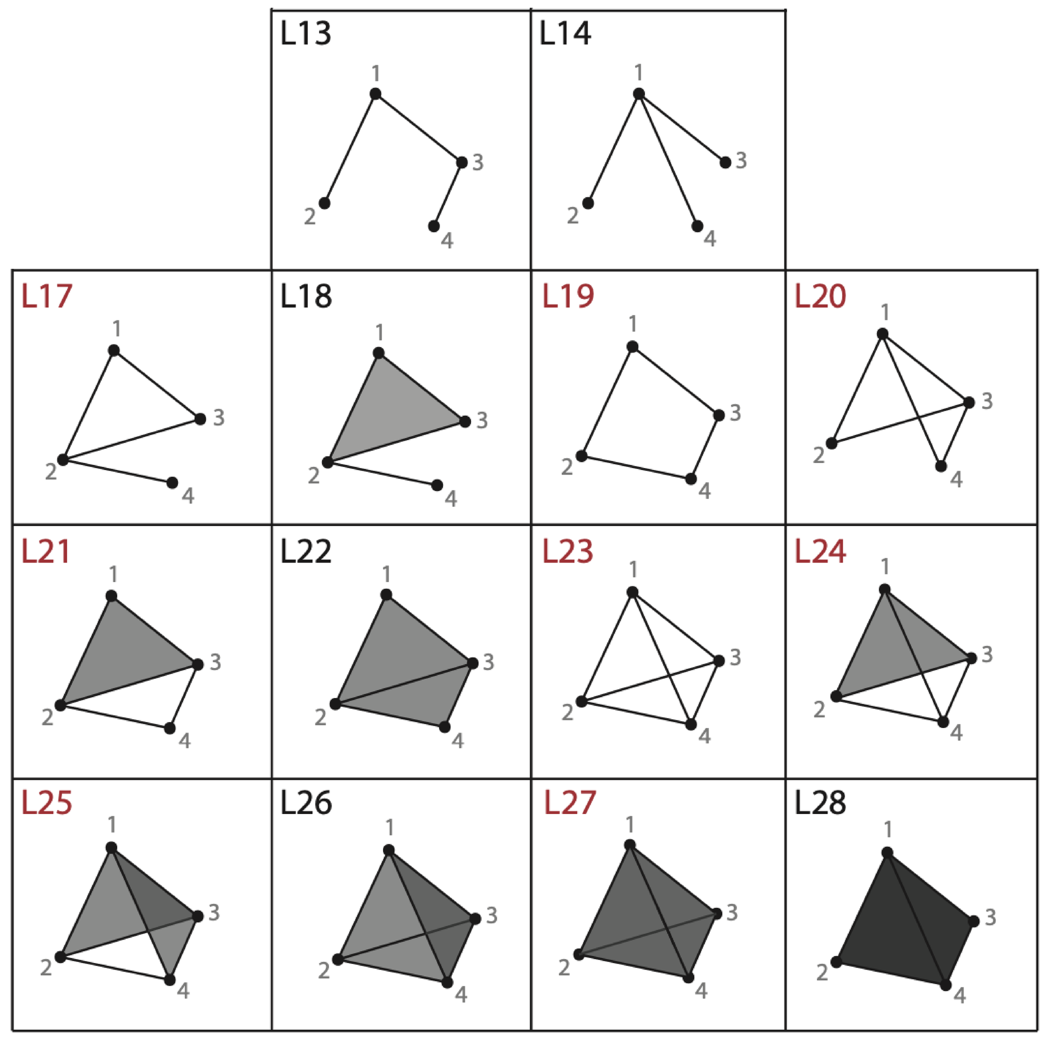


Figure 6: Connected simplicial complexes with 4 vertices [2].

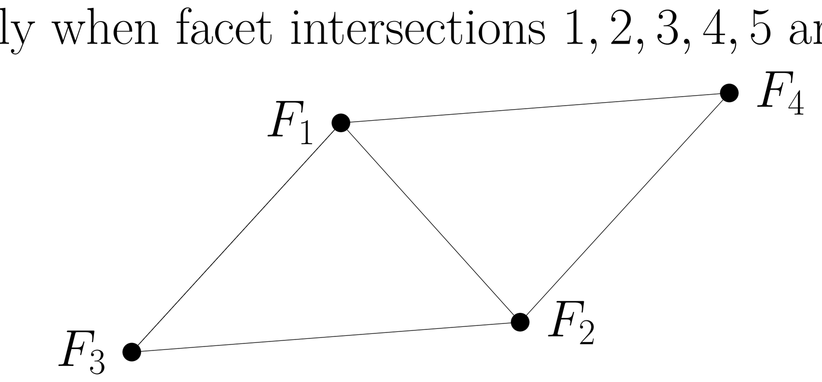
Results

Theorem

Let $\mathcal{C}$ be a 4-maximal code. If $\mathcal{N}(\mathcal{F})$ contains no 2-simplices, then the following are equivalent:

- $\mathcal{C}$ has no local obstructions
- $\mathcal{C}$ is max-∩-complete
- $\mathcal{C}$ is convex.

Example: If $\mathcal{C} = \{\mathbf{123}, \mathbf{145}, \mathbf{24}, \mathbf{35}, \sigma_1, \ldots, \sigma_k\}$, $\mathcal{N}(\mathcal{F})$ does not contain 2-simplices. Thus $\mathcal{C}$ is convex exactly when facet intersections 1, 2, 3, 4, 5 are in $\mathcal{C}$.



Results Continued

Theorem

Let $\mathcal{C}$ be a 4-maximal code. If $\mathcal{N}(\mathcal{F})$ is the simplicial complex L18 or L21, then $\mathcal{C}$ is convex if and only if it contains no local obstructions.

Future Work

We believe our current results can be extended to the L22 case. However, we expect wheels to play a role in cases L24 through L28.

Acknowledgements

We thank the MSRI-UP 2022 program coordinators, namely Federico Ardila, our faculty mentor Anne Shiu, and our research assistant Saber Ahmed. We also thank the MSRI community, the Alfred P. Sloan Foundation, and the National Science Foundation.

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