Tiling Deficient Squares With L– and T– Tetrominoes

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Abstract

We examine a subset of the tetrominoes that includes all rotational and reflective symmetries of L- and T-tetrominoes, and square regions with one square missing, called deficient square regions. We show that all deficient square regions of size $n \times n$ are tileable when $n \geq 7$ and n is odd. We also show that some cases of deficient 3×3 and 5×5 regions are untileable with the chosen tile set.

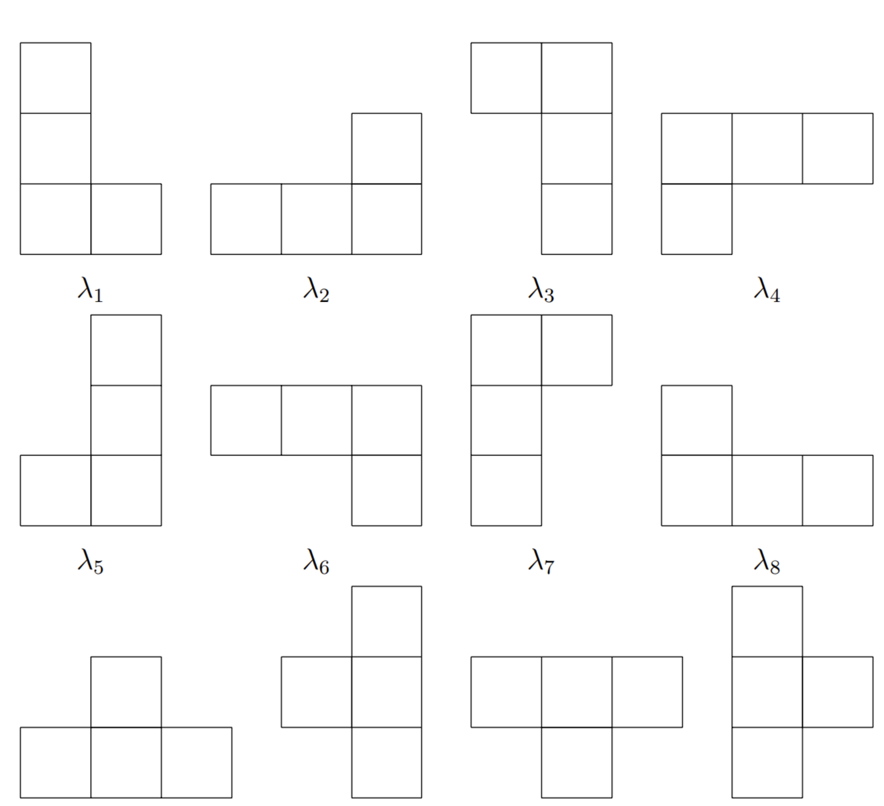
Background Information

Mathematical tiling is a topic in recreational mathematics that explores the possibility of whether polyominoes can tile regions. Polyomino sets are distinguished by their order, or how many squares are joined edge to edge to form each tile in the set. Our tiles may be rotated by multiples of $90°$ and/or reflected along their vertical or horizontal axes. Each of these modifications yields a symmetry of the original tile.

We explore a subset of the order four polyominoes, known as tetrominoes — specifically all rotations and symmetries of L-tetrominoes, which we denote λ_x, and T-tetrominoes denoted τ_x, shown in the Figure below.

The tiles will be used to tile a deficient square region. A region is a group of 1×1 squares grouped together, so a square region is a region with equal sides. The term 'deficient' means that the region is missing one of its 1×1 squares. In other words, no tile in a tiling of the region can cover the missing square. We explore the positions of the square that is removed to determine the places where a deficiency will result in the region being untileable. A past paper [1] by V. Nitica explored tiling deficient rectangles with exclusively L-tetrominoes.

Tileset T



A Size Restriction for the Regions

All of the tiles in the tile set are composed of four squares, so if a region is to be tiled by T with no holes, overlap, or protrusions, then its total number of squares must be divisible by 4. Hence there is a restriction on the size of the square regions we can tile. The area of the region, given by $(n \times n) - 1$ must be divisible by 4. This is the square of the side length minus one unit to account for the deficiency in the region. This can also be written as $n^2 - 1 \equiv 0 \pmod 4$. This is equivalent to saying that n must be odd for the deficient region to be tileable.

Methods

- **Direct Proof:** a short and direct proof by example, used for the 7×7 and 9×9 regions
- **Symmetry:** symmetry of the tile set and region means a finding holds for all reflections and $90°$ rotations
- **Intuitive tiling:** we place a tile and follow through with the forced placement or next most prudent tile placement, used for the 3×3 and 5×5 regions
- **Strong induction:** a mathematical proof technique in which a base case, $P(0)$, is proved to be true, and then an induction step is proved to be true. The induction step proves $P(k+1)$ is true when $P(1), P(2), \cdots, P(k)$ are true

A direct proof is used to show that 7×7 and 9×9 deficient regions are tileable no matter where the deficiency exists. A strong induction is used with a deficient 7×7 and a 9×9 square in the base case to prove that all $n \times n$ deficient square regions are tileable when $n \geq 7$ and n is odd.

Untileability of a Deficient 3×3 Region

Consider the corner case. An example of this case is shown in the figure below. The placement of any tile from the set will cause an untileability. There are eight squares to fill, that means there will be two tiles to fill it. Place any tile, anywhere, in any orientation, and the four remaining squares to be filled are in the shape of a skew tetromino, a square tetromino, or they are separated from each other. All of these are untileable in the chosen tile set. An edge deficiency can be tiled with an L-tetromino and a T-tetromino and the center deficiency can be tiled with two L-tetrominoes as shown in Figure 2.

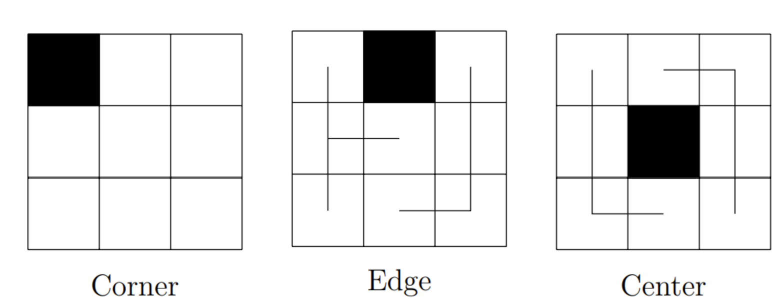


Figure 2: Corner, center and edge deficiencies in a 3×3 region

Untileability of a Deficient 5×5 Region

Not all 5×5 deficient regions can be tiled by T. Figure 3 shows in grey the places where the deficiency can exist while the region is still able to be tiled.

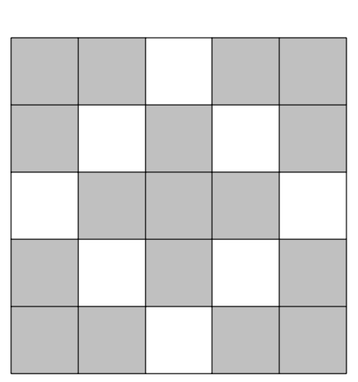


Figure 3: Tileable deficiencies (in grey)

The region cannot be tiled when the deficiency is in one of the white squares. This is proved by exhaustion of all the possible tile arrangements. Since it is only a 5×5 board and the places where the deficiency could be are symmetric, there are not very many cases to check. In fact, one only needs check all possible tile arrangements of two deficiencies — one in the corner of the middle layer and one on the edge of the outer layer. Then by a series of $90°$ rotations all the rest of the cases are accounted for.

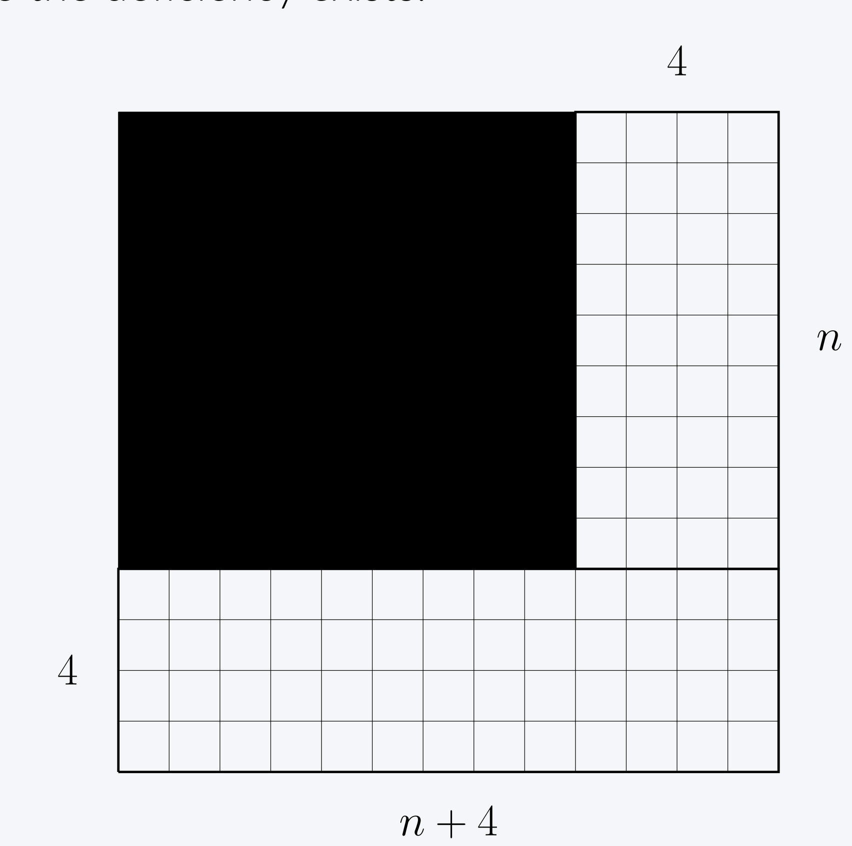
All $n \times n$ Regions where $n \geq 7$ and n is Odd

7×7 and 9×9 Cases

As it turns out, it doesn't matter where the deficiency is in a 7×7 or 9×9 region. The critical sub-region is just the $\frac{n+1}{2} \times \frac{n+1}{2}$ square in the top right of the region. For a region to be tileable for any deficiency, one just has to show that all the deficiencies in the critical sub-region are tileable, and then symmetries mean the deficiency can be anywhere in the whole region. Direct proof shows that there exists at least one possible tiling for every possible deficiency in regions of these sizes.

The Jump to All $n \times n$ Regions

The fact that the $n = 7$ and $n = 9$ cases are tileable is the base case for a strong induction. The critical sub-region of an $(n+4) \times (n+4)$ region is always contained within an $n \times n$ region if the upper left corners of the regions are aligned. That means that since a 7×7 region can be tiled, so can the critical sub-region of an 11×11 region. Then the remaining parts can be tiled by combinations of 2×4 blocks (made from tiles λ_1 and λ_3) and 3×4 blocks (made from λ_1, τ_2, and τ_3). The same goes for a 9×9 region: it's tileability for any deficiency means a 13×13 region is also tileable for any deficiency. The pattern continues *ad infinitum* so that every $n \times n$ deficient square region when $n \geq 7$ and n is odd can be tiled no matter where the deficiency exists.



Conclusion

Using tileset T, an $n \times n$ deficient square region can always be tiled, no matter where the deficiency is, as long as it is of size 7×7 or larger and $n^2 - 1 \equiv 0 \pmod 4$. There are places in a 5×5 square region where a deficiency will cause the region to be untileable. A 3×3 deficient square region can be tiled if and only if the deficiency is not in a corner.

Special Thanks

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References

[1] Viorel Nitica. Tiling a deficient rectangle by l-tetrominoes. *J. Recreational Mathematics*, 33(4):259–271, 2007.