

Abstract

We consider three dimensional iterations of the L-tetrominoe. We show that three dimensional rectangles missing one square, called deficient rectangles, can be tiled as long as they satisfy two conditions: minimum side length 3, and all sides are congruent to 1 (mod 4) or two sides that are congruent to 3 (mod 4) and one side is 1 (mod 4).

Background

Mathematicians try to determine what kinds of regions are “tileable” (fully coverable with no tiles overlapping) given a certain set of tiles. We determine when a particular set of three dimensional (3D) tiles can tile a 3D region. Furthermore, we are concerned mainly with the tiling of three dimensional “deficient” regions. A *deficient region* is a region with one square removed.

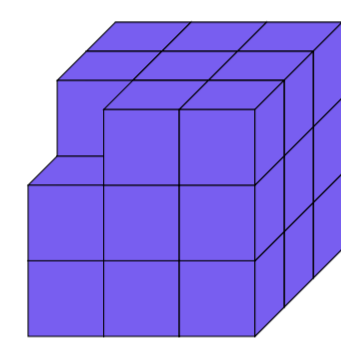


Figure 1. A $3 \times 3 \times 3$ deficient rectangle

Two rectangular regions of the same dimension are called *similar* if one is a rotation of the other. Given a region is tileable by \mathcal{L}^2 or \mathcal{L}^3 , then the tile set can tile all regions similar to it as well.

The Tile Set

For the tile set \mathcal{L}^2 of 2D L-tetrominoes, we define a single tile L^2 which will act as the generator for the tile set. Similarly, for the tile set \mathcal{L}^3 of 3D L-tetrominoes, we define a single tile L^3 which will act as the generator for the tile set. The character L^3 will also be used to refer to any element of \mathcal{L}^3 .



Figure 2. Generators of the Tile Sets \mathcal{L}^2 and \mathcal{L}^3

We define the tile set \mathcal{L}^2 as all combinations of 90° rotations about the origin and reflections about the y axis within \mathbb{Z}^2 of this generating tile. Similarly, we define \mathcal{L}^3 as the set of all possible combinations of 90° rotations about the origin and reflections about any of the three coordinate axes in \mathbb{Z}^3 of this generating tile.

Visual Proof Conventions

Unless otherwise noted, when showing a visual proof for a tiling, we will use a “top-down” view as shown in the figure below:



Figure 3. “Top Down View” Approach

The Corner Region $\mathcal{C}_{(3,4)}^3$

An important region in tilings with \mathcal{L}^3 is a “corner” region. A *corner region*, denoted $\mathcal{C}_{(b,c)}$, is the resulting region after a $(b-1) \times (b-1) \times c$ rectangle is removed from one corner of a $b \times b \times c$ region.

The region $\mathcal{C}_{(3,4)}^3$ is a corner region in \mathbb{Z}^3 with length 3 and height 4. The $\mathcal{C}_{(3,4)}^3$ can be tiled by \mathcal{L}^3 as shown in Figure 4. Note that the figure on the left denotes the tiling in its entirety, while the figure on the right is decomposed to show the tiling more clearly.



Figure 4. The tile set \mathcal{L}^3 tiles $\mathcal{C}_{(3,4)}^3$

Tiling the Region $x \times y \times 4z$

We begin with $z = 1$ and address cases where $z > 1$ at the end of the proof.

Proof.

Case 1) x is even and y is either even or odd: Since x is even, we can write it as $x = 2n$. Hence, we can write the region as $2n \times y \times 4$. This region can be decomposed into ny rectangles of the form $2 \times 1 \times 4$ which are tileable by \mathcal{L}^3 since $2 \times 1 \times 4$ is similar to $2 \times 4 \times 1$ which is tileable by \mathcal{L}^3 because the 2×4 is tileable by \mathcal{L}^2 as shown by Nitica in [4].

Case 2) x and y are odd: Here, $x \times y \times 4$ can be written as $(2n+1) \times (2m+1) \times 4$. We can decompose this rectangle into a $\mathcal{C}_{(3,4)}^3$, a $2n \times 2m \times 4$, and $(m-1) \times (n-1)$ rectangles similar to a $2 \times 4 \times 1$, all of which are tileable by \mathcal{L}^3 .

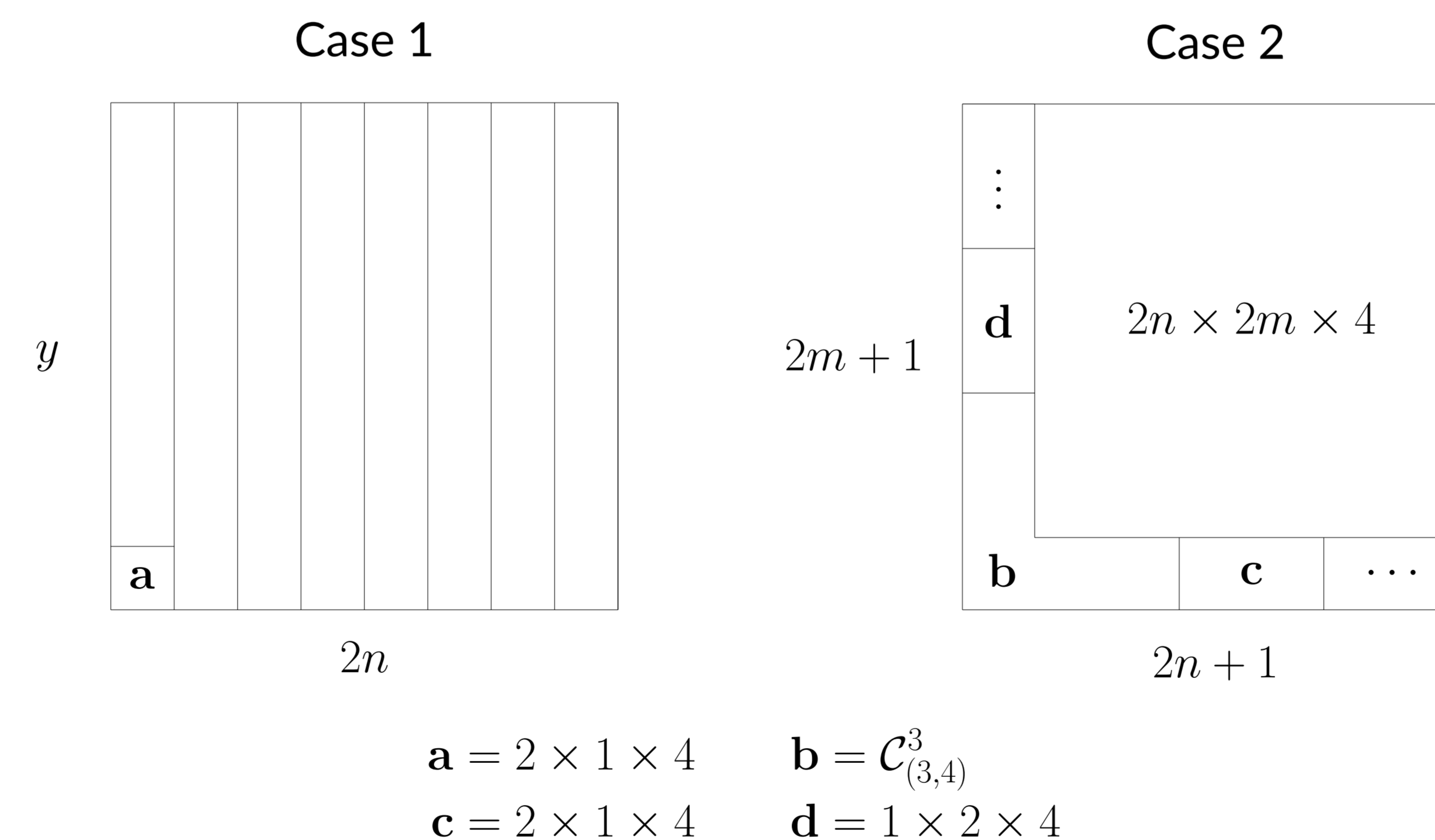


Figure 5. Decomposition of $2n \times y \times 4$

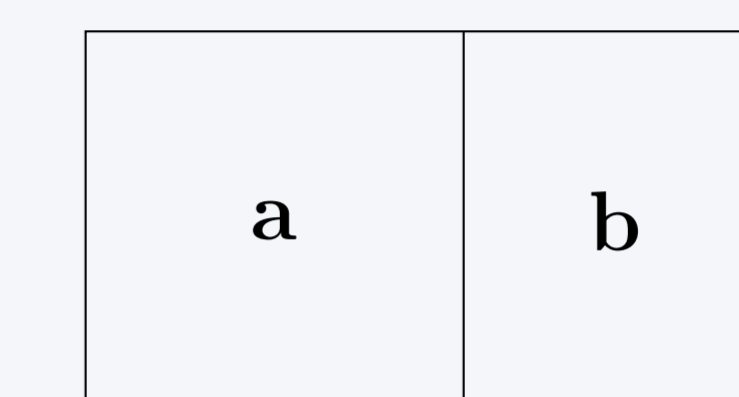
Since \mathcal{L}^3 can tile any region of the form $x \times y \times 4$, it can tile any region of the form $x \times y \times 4z$ as it is just z copies of $x \times y \times 4$. □

Deficient Tilings of Rectangular Candidates

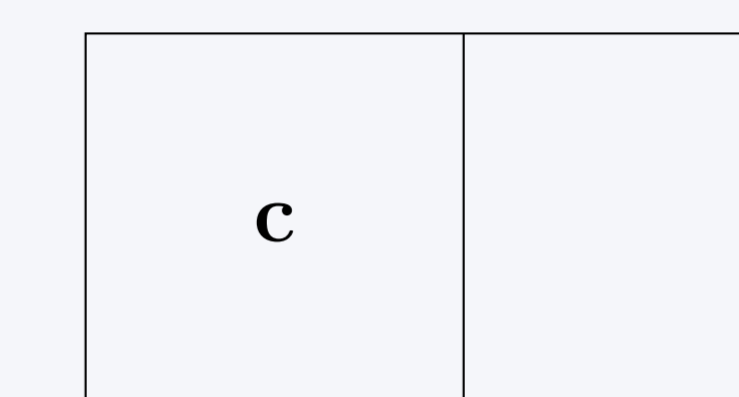
In order to be tiled, a deficient rectangle must have area $4n-1$, where $n \geq 2$. This is called the *area invariant*. Rectangles that satisfy this invariant either have the form $(4x+1) \times (4y+1) \times (4z+1)$ or $(4x+3) \times (4y+1) \times (4z+3)$. We will address the only the former tiling in this poster.

Proof.

Case 1) $n > m$: Our strategy is to tile $(4m+1) \times (4m+1) \times 1$ deficiently on the left side. There are z regions of the form $(4m+1) \times (4m+1) \times 4$ above the deficiently tiled region which are fully tileable as discussed in the previous section. To the right, we tile the remaining $(n-m)$ regions of the form $4 \times (4m+1) \times (4z+1)$ fully.



$z = 0$



$z = 1$

$$a = ((4m+1) \times (4m+1) \times 1)_D$$

$$b \text{ is } (n-m) \text{ copies of } (4 \times (4m+1) \times 5)$$

$$c = (4m+1) \times (4m+1) \times 4z$$

Figure 6. The tile set \mathcal{L}^3 tiles $(4n+1) \times (4m+1) \times 5$ with $n > m$.

Case 2) $m > n$: By symmetry, this is Case 1.

Case 3) $m = n$: Nitica showed in [4] that any region $(4n+1) \times (4n+1) \times 1$ is deficiently tileable by \mathcal{L}^3 . Therefore, we begin by tiling $(4n+1) \times (4n+1) \times 1$ deficiently. The $(4n+1) \times (4n+1) \times 4z$ region that lies above the deficiently tiled region is tileable as discussed in the previous section. □

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